

# Stability Margins of $\mathcal{L}_1$ Adaptive Controller: Part II<sup>\*</sup>

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## Abstract

In Part I of this paper, [1], we have developed a novel  $\mathcal{L}_1$  adaptive control architecture that enables fast adaptation and leads to uniformly bounded transient and asymptotic tracking for system's both signals, input and output, simultaneously. In this paper, we derive the stability margins of  $\mathcal{L}_1$  adaptive control architecture, including time-delay and gain margins in the presence of time-varying bounded disturbance. Simulations verify the theoretical findings.

## 1 Introduction

Adaptive control schemes have proven to be extremely useful in a number of flight tests for recovering the nominal performance in the presence of modeling and environmental uncertainties (see [2] and references therein). A major challenge in analysis of these systems is determining its stability margins dependent upon the adaptation gain. Today it largely relies on the numerical evidence provided by Monte-Carlo schemes. It has been observed that increasing the adaptation gain leads to improved tracking performance, but results in high-frequency oscillations in the control signal and reduces the system's tolerance to the time-delay in the control and the sensor channels.

In the linear time invariant (LTI) systems theory, stability margins are defined by the gain and the phase margins. Phase margin characterizes the amount of additional phase lag at the gain-crossover frequency required to bring the system to the verge of instability. Phase margin is significant in predicting how much time-delay the system can endure in its input/output channels before it loses its stability.

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While the gain margin can be generalized for nonlinear systems, the notion of the phase margin cannot be extended to nonlinear systems in straightforward manner. Instead it is common to use sector and disk margins for nonlinear systems [3]. However, from practical control design perspective these notions are not as useful as the phase margin in the linear systems theory. In this paper, instead of the phase margin we introduce the notion of the time-delay margin directly for the closed-loop nonlinear adaptive system. Time-delay margin characterizes the maximum time-delay in the (sensor) channel that the closed-loop system can tolerate before it loses its stability. In linear systems theory this corresponds to the ratio of the phase margin to the cross-over frequency of its Bode plot. Similarly, the gain margin is the maximum open loop gain that the closed-loop system can tolerate before it loses its stability.

In [4, 5], we have introduced novel  $\mathcal{L}_1$  adaptive control architecture that has guaranteed transient performance. In [1], we have extended the approach to systems with unknown time-varying parameters and bounded disturbances. In this paper, we derive the stability margins for the  $\mathcal{L}_1$  adaptive control architecture from [1], which we specialize for unknown constant parameters and bounded time-varying disturbances. While the analysis of the gain-margin is relatively straightforward, the analysis of its time-delay margin takes several steps. At first we introduce an equivalent linear-time invariant (LTI) system, subject to an exogenous input dependent upon the parameters and time trajectories of certain signals of the closed-loop adaptive system. We prove that with the same initial conditions in the presence of the same time-delay in the output channels of these two systems there exists at least one exogenous input such that their resulting trajectories are the same over the entire time-horizon. Next, we prove that for every value of the time-delay within the time-delay margin of this LTI system there exists a lower bound for the adaptive gain that renders this exogenous input bounded.

We notice that characterization of the time-delay margin is extremely difficult as compared to the gain-margin analysis for nonlinear closed-loop systems. To the best of our knowledge there are no such results in adaptive control theory, despite the fact that there is a large body of well-established literature on adaptive control of time-delay systems. Control of time-delay systems and determining the time-delay margin of a closed-loop system are principally different problems, and one cannot be used to provide a solution for the other. On the other hand, this is not surprising since the time-delay margin cannot be characterized if the transient is not guaranteed.

The paper is organized as follows. Section 2 states some preliminary definitions, and Section 3 gives the problem formulation. In Section 4, the  $\mathcal{L}_1$  adaptive controller is presented. Stability margins, including time-delay and gain margins, are derived in Section 5. Results of [1] and of this paper are generalized in Section

8. In section 9, simulation results are presented, while Section 10 concludes the paper. The proof of the main theorem is in Appendix.

## 2 Preliminaries

In this Section, we recall some basic definitions and facts from linear systems theory, [6–8].

**Definition 1:** For a signal  $\xi(t) = [\xi_1(t), \dots, \xi_n(t)]^\top$ ,  $t \geq 0$ , its truncated  $\mathcal{L}_\infty$  norm and  $\mathcal{L}_\infty$  norm are defined as  $\|\xi_t\|_{\mathcal{L}_\infty} = \max_{i=1,\dots,n}(\sup_{0 \leq \tau \leq t} |\xi_i(\tau)|)$ ,  $\|\xi\|_{\mathcal{L}_\infty} = \max_{i=1,\dots,n}(\sup_{\tau \geq 0} |\xi_i(\tau)|)$ .

**Definition 2:** The  $\mathcal{L}_1$  gain of a stable proper single-input single-output system  $H(s)$  is defined as  $\|H(s)\|_{\mathcal{L}_1} = \int_0^\infty |h(t)|dt$ , where  $h(t)$  is the impulse response of  $H(s)$ , computed via the inverse Laplace transform  $h(t) = \frac{1}{2\pi i} \int_{\alpha-i\infty}^{\alpha+i\infty} H(s)e^{st}ds$ ,  $t \geq 0$ , in which integration is done along the vertical line  $x = \alpha > 0$  in complex plane.

*Proposition:* A continuous time LTI system (proper) with impulse response  $h(t)$  is stable if and only if  $\int_0^\infty |h(\tau)|d\tau < \infty$ . A proof can be found in [6] (page 81, Theorem 3.3.2).

**Definition 3:** For a stable proper  $m$  input  $n$  output system  $H(s)$  its  $\mathcal{L}_1$  gain is defined as  $\|H(s)\|_{\mathcal{L}_1} = \max_{i=1,\dots,n} \left( \sum_{j=1}^m \|H_{ij}(s)\|_{\mathcal{L}_1} \right)$ , where  $H_{ij}(s)$  is the  $i^{th}$  row  $j^{th}$  column element of  $H(s)$ .

**Lemma 1:** For a stable proper multi-input multi-output (MIMO) system  $H(s)$  with input  $r(t) \in \mathbb{R}^m$  and output  $x(t) \in \mathbb{R}^n$ , we have  $\|x_t\|_{\mathcal{L}_\infty} \leq \|H\|_{\mathcal{L}_1} \|r_t\|_{\mathcal{L}_\infty}$ ,  $\forall t \geq 0$ .

**Corollary 1:** For a stable proper MIMO system  $H(s)$ , if the input  $r(t) \in \mathbb{R}^m$  is bounded, then the output  $x(t) \in \mathbb{R}^n$  is also bounded as  $\|x\|_{\mathcal{L}_\infty} \leq \|H(s)\|_{\mathcal{L}_1} \|r\|_{\mathcal{L}_\infty}$ .

Consider a linear time invariant system:  $\dot{x}(t) = Ax(t) + bu(t)$ , where  $x \in \mathbb{R}^n$ ,  $u \in \mathbb{R}$ ,  $b \in \mathbb{R}^n$ ,  $A \in \mathbb{R}^{n \times n}$  is Hurwitz, and assume  $(sI - A)^{-1}b$  is strictly proper and stable. Notice that it can be expressed as:  $(sI - A)^{-1}b = \frac{n(s)}{d(s)}$ , where  $d(s) = \det(sI - A)$  is a  $n^{th}$  order stable polynomial, and  $n(s)$  is a  $n \times 1$  vector with its  $i^{th}$  element being a polynomial function:  $n_i(s) = \sum_{j=1}^n n_{ij}s^{j-1}$ .

**Lemma 2:** If  $(A \in \mathbb{R}^{n \times n}, b \in \mathbb{R}^n)$  is controllable, the matrix  $N$  with its  $i^{th}$  row  $j^{th}$  column entry  $n_{ij}$  is full rank.

**Lemma 3:** If  $(A, b)$  is controllable and  $(sI - A)^{-1}b$  is strictly proper and stable, there exists  $c \in \mathbb{R}^n$  such that  $c^\top(sI - A)^{-1}b$  is minimum phase with relative degree one, i.e. all its zeros are located in the left half plane, and its denominator is one order larger than its numerator.

Also, we introduce the following notations that will be used throughout the paper. Let  $x_h(t)$  be the state variable of the LTI system  $H_x(s)$ , while  $x_i(t)$  and  $x_s(t)$  be the input and the output signals of it. We note that for any time instant  $t_1$  and any fixed time-interval  $[t_1, t_2]$ , where  $t_2 > t_1$ , given  $x_h(t_1)$  and an impulse-free input signal  $x_i(t)$  over  $[t_1, t_2]$ ,  $x_s(t)$  is uniquely defined for  $t \in [t_1, t_2]$ . Let  $\mathcal{S}$  be the map  $x_s(t)|_{t \in [t_1, t_2]} = \mathcal{S}(H_x(s), x_h(t_1), x_i(t)|_{t \in [t_1, t_2]})$ . We note that  $\mathcal{S}$  is continuous, if  $x_i(t)$  is impulse free. Also,  $x_s(t)$  is defined over a closed interval  $[t_1, t_2]$ , although  $x_i(t)$  is defined over the corresponding open set  $[t_1, t_2)$ . The next lemma follows from the definition of  $\mathcal{S}$ .

**Lemma 4:** Let  $x_{o1}|_{t \in [t_1, t_2]} = \mathcal{S}(H_x(s), x_{h1}, x_{i1}(t)|_{t \in [t_1, t_2]})$ ,  $x_{o2}|_{t \in [t_1, t_2]} = \mathcal{S}(H_x(s), x_{h2}, x_{i2}(t)|_{t \in [t_1, t_2]})$ . If  $x_{h1} = x_{h2}$  and  $x_{i1}(t) = x_{i2}(t)$  over  $[t_1, t_2]$ , then  $x_{o1}(t) = x_{o2}(t)$  for any  $t \in [t_1, t_2]$ .

### 3 Problem Formulation

Consider the following single-input single-output system dynamics:

$$\begin{aligned}\dot{x}(t) &= A_m x(t) + b \left( \omega u(t) + \theta^\top x(t) + \sigma(t) \right), x(0) = x_0 \\ y(t) &= c^\top x(t),\end{aligned}\tag{1}$$

where  $x \in \mathbb{R}^n$  is the system state vector (measurable),  $u \in \mathbb{R}$  is control signal,  $y \in \mathbb{R}$  is the regulated output,  $b, c \in \mathbb{R}^n$  are known constant vectors,  $A_m \in \mathbb{R}^{n \times n}$  is given Hurwitz matrix,  $\omega \in \mathbb{R}$  is unknown constant with given sign,  $\theta \in \mathbb{R}^n$  is unknown constant vector, and  $\sigma(t) \in \mathbb{R}$  is a uniformly bounded time-varying disturbance with a uniformly bounded derivative. Without loss of generality, we assume

$$\omega \in \Omega_0 = [\omega_{l_0}, \omega_{u_0}], \theta \in \Theta, |\sigma(t)| \leq \Delta_0, \forall t \geq 0,\tag{2}$$

where  $\omega_{u_0} > \omega_{l_0} > 0$  are known (conservative) upper and lower bounds,  $\Theta$  is a known (conservative) compact set and  $\Delta_0 \in \mathbb{R}^+$  is a known (conservative)  $\mathcal{L}_\infty$  bound of  $\sigma(t)$ . We further assume that  $\sigma(t)$  is continuously differentiable and its derivative is uniformly bounded, i.e.  $|\dot{\sigma}(t)| \leq d_\sigma < \infty$  for any  $t \geq 0$ , where  $d_\sigma$  can be arbitrarily large as long as it is finite.

In [1], we have considered the system in (1) in the presence of time-varying  $\theta(t)$  and have designed an adaptive controller to ensure that  $y(t)$  tracks a given bounded

continuous reference signal  $r(t)$  *both in transient and steady state*, while all other error signals remain bounded. The main result of [1] implies that by increasing the adaptation gain one can get arbitrarily close transient and asymptotic tracking of a desired reference system. In [1], we have also considered the particular case of constant  $\theta$ . Here we investigate the stability margins for this latter case.

#### 4 $\mathcal{L}_1$ Adaptive Controller

In this section, we repeat the  $\mathcal{L}_1$  adaptive control architecture for the system in (1) that permits complete transient characterization for both  $u(t)$  and  $x(t)$ , [1]. The elements of  $\mathcal{L}_1$  adaptive controller are introduced next without repeating the proofs from [1].

**Companion Model:** The companion model is:

$$\begin{aligned}\dot{\hat{x}}(t) &= A_m \hat{x}(t) + b(\hat{\omega}(t)u(t) + \hat{\theta}^\top(t)x(t) + \hat{\sigma}(t)), \\ \hat{y}(t) &= c^\top \hat{x}(t), \quad \hat{x}(0) = x_0,\end{aligned}\tag{3}$$

which has the same dynamic structure as the system in (1). Only the unknown parameters and the disturbance  $\omega, \theta, \sigma(t)$  are replaced by their adaptive estimates  $\hat{\omega}(t), \hat{\theta}(t), \hat{\sigma}(t)$ .

**Adaptive Laws:** Adaptive estimates are governed by the following laws:

$$\dot{\hat{\theta}}(t) = -\Gamma_\theta \text{Proj}(x(t)\tilde{x}^\top(t)Pb, \hat{\theta}(t)),\tag{4}$$

$$\dot{\hat{\sigma}}(t) = -\Gamma_\sigma \text{Proj}(\tilde{x}^\top(t)Pb, \hat{\sigma}(t)),\tag{5}$$

$$\dot{\hat{\omega}}(t) = -\Gamma_\omega \text{Proj}(u(t)\tilde{x}^\top(t)Pb, \hat{\omega}(t)),\tag{6}$$

where  $\tilde{x}(t) = \hat{x}(t) - x(t)$  is the error signal between the state of the system and the companion model,  $P$  is the solution of the algebraic equation  $A_m^\top P + PA_m = -Q$ ,  $Q > 0$ ,  $\Gamma_\theta = \Gamma_c \mathbb{I}_{n \times n} \in \mathbb{R}^{n \times n}$ ,  $\Gamma_\sigma = \Gamma_\omega = \Gamma_c$  are adaptation gains with  $\Gamma_c \in \mathbb{R}^+$ . In the implementation of the projection operator we use the compact sets  $\Theta$  as given in (2), while we replace  $\Delta_0, \Omega_0$  by larger sets  $\Delta$  and  $\Omega = [\omega_l, \omega_u]$  such that

$$\Delta_0 < \Delta, \quad 0 < \omega_l < \omega_{l_0} < \omega_{u_0} < \omega_u.\tag{7}$$

The purpose of this will be shortly clarified in the analysis of the stability margins.

**Control Law:** The control signal is generated through gain feedback of the following system:

$$\chi(s) = D(s)r_u(s), \quad u(s) = -k\chi(s),\tag{8}$$

where  $r_u(s)$  is the Laplace transformation of  $r_u(t) = \hat{\omega}(t)u(t) + \bar{r}(t)$ ,  $k \in \mathbb{R}^+$  is a feedback gain,  $\bar{r}(t) = \hat{\theta}^\top(t)x(t) + \hat{\sigma}(t) - k_g r(t)$ ,  $k_g = -\frac{1}{c^\top A_m^{-1}b}$ , and  $D(s)$  is a LTI system that needs to be chosen to ensure

$$C(s) = \frac{\omega k D(s)}{1 + \omega k D(s)} \quad (9)$$

is stable and strictly proper with  $C(0) = 1$ . One choice is  $D(s) = \frac{1}{s}$ , that leads to  $C(s) = \frac{\omega k}{s + \omega k}$ . Let  $L = \max_{\theta \in \Theta} \sum_{i=1}^n |\theta_i|$ . We now give the  $\mathcal{L}_1$  performance requirement that ensures desired transient performance, [1].

**$\mathcal{L}_1$ -gain stability requirement:** Design  $D(s)$  to ensure that  $C(s)$  in (9) satisfies

$$\|G(s)\|_{\mathcal{L}_1} L < 1, \quad (10)$$

where  $G(s) = H(s)(1 - C(s))$ , and  $H(s) = (sI - A_m)^{-1}b$ .

The complete  $\mathcal{L}_1$  adaptive controller consists of (3), (4)-(6), (8) subject to (10). We notice that the  $\mathcal{L}_1$ -gain stability requirement depends only upon the choice of  $\Theta$  and is independent of the choice of  $\Delta_0$ ,  $\Omega_0$  or  $\Delta$ ,  $\Omega$ .

## 5 Analysis of $\mathcal{L}_1$ Adaptive Controller

Next, consider the following closed-loop reference system with the control signal  $u_{ref}(t)$  and the system response  $x_{ref}(t)$ , the stability of which, subject to (10), can be proven using the small-gain theorem, [1]:

$$\begin{aligned} \dot{x}_{ref}(t) &= A_m x_{ref}(t) + b(\omega u_{ref}(t) + \theta^\top x_{ref}(t) + \sigma(t)) \\ u_{ref}(s) &= C(s) \frac{\bar{r}_{ref}(s)}{\omega}, \quad y_{ref}(t) = c^\top x_{ref}(t), \end{aligned} \quad (11)$$

with  $x_{ref}(0) = x_0$ , where  $\bar{r}_{ref}(s)$  is the Laplace transformation of the signal  $\bar{r}_{ref} = -\theta^\top x_{ref}(t) - \sigma(t) + k_g r(t)$ .

**Lemma 5:** [1] For the system in (1) and the  $\mathcal{L}_1$  adaptive controller in (3), (4)-(6) and (8), we have  $\|\tilde{x}\|_{\mathcal{L}_\infty} \leq \sqrt{\frac{\theta_m}{\lambda_{\min}(P)\Gamma_c}}$  where  $\theta_m \triangleq \max_{\theta \in \Theta} \sum_{i=1}^n 4\theta_i^2 + 4\Delta^2 + 4(\omega_u - \omega_l)^2 + 2\frac{\lambda_{\max}(P)}{\lambda_{\min}(Q)}b_\sigma\Delta$ .

Lemma 3 ensures existence of  $c_o \in \mathbb{R}^n$  such that  $c_o^\top H(s) = \frac{N_n(s)}{N_d(s)}$ , where the order of  $N_d(s)$  is one more than the order of  $N_n(s)$ , and both  $N_n(s)$  and  $N_d(s)$  are stable polynomials.

**Theorem 1:** [1] Given the system in (1) and the  $\mathcal{L}_1$  adaptive controller defined via (3), (4)-(6) and (8) subject to (10), we have:  $\|x - x_{ref}\|_{\mathcal{L}_\infty} \leq \gamma_1$ ,  $\|u - u_{ref}\|_{\mathcal{L}_\infty} \leq \gamma_2$ , where  $\gamma_1 = \frac{\|C(s)\|_{\mathcal{L}_1}}{1 - \|G(s)\|_{\mathcal{L}_1} L} \sqrt{\frac{\theta_m}{\lambda_{\max}(P)\Gamma_c}}$ , and  $\gamma_2 = \left\| \frac{C(s)}{\omega} \theta^\top \right\|_{\mathcal{L}_1} \gamma_1 + \left\| \frac{C(s)}{\omega} \frac{1}{c_o^\top H(s)} c_o^\top \right\|_{\mathcal{L}_1} \sqrt{\frac{\theta_m}{\lambda_{\max}(P)\Gamma_c}}$ .

## 6 Time-delay Margin Analysis

### 6.1 $\mathcal{L}_1$ adaptive controller in the presence of time-delay

To analyze the time-delay margin of the closed-loop adaptive system in the next section we consider a linear time-invariant (LTI) system subject to an external exogenous input. We develop sufficient conditions under which that LTI system can be used to evaluate the time-delay margin of the closed-loop adaptive system. Before then, we need to introduce the following three systems.

**System 1.** We rewrite the open-loop system in (1) as

$$x(s) = \bar{H}(s)(\omega u(s) + \sigma(s)), \quad (12)$$

where  $\bar{H}(s) = (sI - A_m - b\theta^\top)^{-1}b$ . Without loss of generality, we set:

$$x(0) = 0. \quad (13)$$

Let  $x_d(t)$  be the delayed signal of the open-loop state  $x(t)$  by a constant time interval  $\tau$ , i.e

$$x_d(t) = \begin{cases} x(t - \tau) & t \geq \tau, \\ 0 & t < \tau. \end{cases} \quad (14)$$

We close the loop of (12) with  $\mathcal{L}_1$  adaptive controller (3), (4)-(6), (8), using  $x_d(t)$  from (14) instead of  $x(t)$  everywhere in the definition of (3), (4)-(6), (8). We denote the resulting control and state trajectories of this closed-loop system by  $u(t)$  and  $x_d(t)$ . We further notice that this closed-loop adaptive system has a unique solution. It is the stability of this closed-loop system that we are investigating in this paper, dependent upon  $\tau$ . It is important to point out that while applying the  $\mathcal{L}_1$  adaptive controller (3), (4)-(6), (8) to the system in (12) using  $x_d(t)$  from (14), one cannot derive the dynamics of the error signal between the system state and the companion model, the boundedness of which is stated in Lemma 5. Neither Theorem 1 is valid.

**System 2.** Next, we consider the following closed-loop system with the same zero initial conditions:

$$\dot{x}_q(t) = A_m x_q(t) + b \left( \omega u_q(t) + \theta^\top x_q(t) + \sigma(t) + \eta(t) \right), \quad (15)$$

where  $x_q(0) = x(0)$ ,  $\theta$  and  $\sigma(t)$  have been introduced in (1),  $u_q(t)$  is defined via (3), (4)-(6) and (8), while  $\eta(t)$  is a continuously differentiable bounded signal with uniformly bounded derivative. As compared to (1) or (12), the system in (15) has one more additional disturbance signal  $\eta(t)$ . If

$$|\sigma(t) + \eta(t)| \leq \Delta, \quad (16)$$

where  $\Delta$  has been defined in (7), then application of  $\mathcal{L}_1$  adaptive controller to the system in (15) is well defined, and hence the results of Theorem 1 are valid for the system in (15) as well. We denote by  $u_q(t)$  the time trajectory of the  $\mathcal{L}_1$  adaptive controller, resulting from its application to (15).

**System 3.** Finally, we consider the open-loop system in (12)-(14) and apply  $u_q(t)$  to it and look at its delayed output  $x_o(t)$ , where the subindex  $o$  is added to indicate the open-loop nature of this signal. It is important to notice that at this point we view  $u_q(t)$  as a time-varying input signal for (12), and not as a feedback signal, so that (12) remains an open-loop system in this context.

Illustration of these last two systems is given in Fig. 1.

Fig. 1: Systems 2 and 3

**Lemma 6:** If the time-delayed output of the open-loop System 3 has the same time history as the closed-loop output of System 2, i.e.

$$x_o(t) = x_q(t), \quad \forall t \geq 0, \quad (17)$$

then  $u(t) = u_q(t)$ ,  $x_d(t) = x_q(t)$ ,  $\forall t \geq 0$ , where  $u(t)$  and  $x_d(t)$  denote the control and state trajectories of the closed-loop System 1 in (12)-(14) with  $\mathcal{L}_1$  adaptive controller.



**Proof.** It follows from (17) that the open-loop time-delayed System 3 in (12)-(14) generates  $x_q(t)$  in response to the input  $u_q(t)$ . When applied to (15),  $u_q(t)$  leads to  $x_q(t)$ . Hence,  $u_q(t)$  and  $x_q(t)$  are also solutions of the closed-loop adaptive System 1 in (12)-(14) with (3), (4)-(6), (8).  $\square$

This Lemma consequently implies that to ensure stability of the System 1 in the presence of a given time-delay  $\tau$ , it is sufficient to prove existence of  $\eta(t)$  in System 2, satisfying (16) and verifying (17). We notice, however, that the closed-loop System 2 is a nonlinear system due to the nonlinear adaptive laws, so that the proof on existence of such  $\eta(t)$  for this system and explicit construction of the set  $\Delta$  is not straightforward. Moreover, we note that the condition in (17) relates the time-delay  $\tau$  of System 1 (or System 3) to the signal  $\eta(t)$  implicitly. In the next section of this paper we introduce an equivalent LTI system that helps to prove existence of such  $\eta(t)$  and leads to explicit construction of  $\Delta$ . Definition of this LTI system is the key step in the overall analysis. It has an exogenous input that lumps the time trajectories of the nonlinear elements of the closed-loop System 2. For this LTI system, the time delay margin can be computed via its open-loop transfer function, which consequently defines a conservative lower bound for the time-delay margin of the adaptive system.

## 6.2 LTI System in the Presence of Time-delay in its Output

Consider the following closed-loop LTI system:

$$\begin{aligned} x_l(s) &= \bar{H}(s)\zeta_l(s), \quad \epsilon_l(s) = (C(s)/\omega)\tilde{r}_l(s) \\ u_l(s) &= (1/\omega)C(s)(k_g r(s) - \theta^\top x_l(s) - \sigma(s) - \eta_l(s)) - \epsilon_l(s) \end{aligned}$$

where  $\zeta_l(s) = \omega u_l(s) + \sigma(s)$ ,  $\eta_l(s) = \zeta_l(s) - \omega u_l(s) - \sigma(s)$ ,  $r(s)$  and  $\sigma(s)$  are the Laplace transformations of the bounded signals  $r(t)$  and  $\sigma(t)$ , respectively,  $x_l(t)$ ,  $u_l(t)$  and  $\epsilon_l(t)$  are selected states,  $\zeta_l(t)$  is its output signal, and  $\tilde{r}_l(s)$  is the Laplace transformation of an exogenous signal  $\tilde{r}_l(t)$ . We note that the system trajectories are uniquely defined once  $\tilde{r}_l(t)$  is given.

We notice that the transfer functions from  $\sigma(t)$  and  $r(t)$  to  $x_l(t)$  are the same as in the reference system. Since  $x_l(s) = \bar{H}(s)\zeta_l(s)$ , we have

$$x_l(s)/r(s) = (\bar{H}(s)C(s))/(1 + C(s)\theta^\top \bar{H}(s)), \quad (18)$$

$$x_l(s)/\sigma(s) = (\bar{H}(s)(1 - C(s)))/(1 + C(s)\theta^\top \bar{H}(s)). \quad (19)$$

One can verify that for the reference system in (11), we have  $x_{ref}(s)/r(s)$  and  $x_{ref}(s)/\sigma(s)$  equivalent to (18) and (19). We also notice that the LTI system without time-delay ensures stable transfer functions from inputs  $r(t)$ ,  $\sigma(t)$  and  $\tilde{r}_l(t)$  to output  $\zeta_l(t)$ .

Assume the system output  $\zeta_l(t)$  experiences time-delay  $\tau$ , so that in the presence of the time-delay we have:

$$x_l(s) = \bar{H}(s)\zeta_{l_d}(s) \quad (20)$$

$$u_l(s) = (C(s)/\omega)(k_g r(s) - \theta^\top x_l(s) - \sigma(s) - \eta_l(s)) - \epsilon_l(s) \quad (21)$$

$$\epsilon_l(s) = (C(s)/\omega)\tilde{r}_l(s) \quad (22)$$

$$\zeta_l(s) = \omega u_l(s) + \sigma(s), \quad (23)$$

where  $\zeta_{l_d}(t)$  is the time-delayed signal of  $\zeta_l(t)$ , i.e

$$\zeta_{l_d}(t) = \begin{cases} 0 & t < \tau, \\ \zeta_l(t - \tau) & t \geq \tau, \end{cases} \quad (24)$$

consequently leading to redefined  $\eta_l(s)$ :

$$\eta_l(s) = \zeta_{l_d}(s) - \omega u_l(s) - \sigma(s). \quad (25)$$

Let

$$x_l(0) = 0, \quad u_l(0) = 0, \quad \epsilon_l(0) = 0. \quad (26)$$

We notice that the system in (20)-(23) is highly coupled. Its diagram is plotted in Figure 2.

Fig. 2: LTI system

### 6.3 Time-Delay Margin of the LTI System

We notice that the phase margin of this LTI system can be determined by its open-loop transfer function from  $\zeta_{l_d}(t)$  to  $\zeta_l(t)$ . It follows from (20), (21), and (25)

that  $\omega u_l(s) = \frac{C(s)(k_g r(s) - \zeta_{l_d}(s) - \theta^\top \bar{H}(s) \zeta_{l_d}(s)) - \omega \epsilon_l(s)}{1 - C(s)}$ , and hence (23) implies that  $\zeta_l(s) = \frac{C(s)(k_g r(s) - \zeta_{l_d}(s) - \theta^\top \bar{H}(s) \zeta_{l_d}(s)) - \omega \epsilon_l(s)}{1 - C(s)} + \sigma(s)$ . Therefore, it can be equivalently written as:

$$\begin{aligned}\zeta_l(s) &= \frac{1}{1 - C(s)} (r_b(s) - r_f(s)) , \\ r_f(s) &= C(s)(1 + \theta^\top \bar{H}(s)) \zeta_{l_d}(s) , \\ r_b(s) &= C(s)k_g r(s) + (1 - C(s))\sigma(s) - \omega \epsilon_l(s) .\end{aligned}\tag{27}$$

Assume that  $\tilde{r}_l(t)$  is such that  $\epsilon_l(t)$  is bounded. Since  $\sigma(t)$  and  $r(t)$  are bounded,  $C(s)$  is strictly proper and stable, then  $r_b(t)$  is also bounded. The block-diagram of the closed-loop system in (27) is shown in Figure 3.

Fig. 3: LTI system

The open-loop transfer function of the system in (27) is:

$$H_o(s) = C(s)(1 + \theta^\top \bar{H}(s))/(1 - C(s)) ,\tag{28}$$

the phase margin  $\mathcal{P}(H_o(s))$  of which can be derived from its Bode plot easily. Its time-delay margin is given by:

$$\mathcal{T}(H_o(s)) = \mathcal{P}(H_o(s))/\omega_c ,\tag{29}$$

where  $\mathcal{P}(H_o(s))$  is the phase margin of the open-loop system  $H_o(s)$ , and  $\omega_c$  is the cross-over frequency of  $H_o(s)$ . The next lemma states sufficient condition for boundedness of all the states in the system (20)-(23), including the internal states.

**Lemma 7:** Let

$$\tau < \mathcal{T}(H_o(s))\tag{30}$$

and  $\epsilon_b$  be any positive number such that  $\|\epsilon_l\|_{\mathcal{L}_\infty} \leq \epsilon_b$ . Then the signals  $\zeta_l(t)$ ,  $x_l(t)$ ,  $u_l(t)$ ,  $\eta_l(t)$  are bounded.

**Proof:** Since  $\epsilon_l(t)$  is bounded and  $\tau < \mathcal{T}(H_o(s))$ , then boundedness of  $\zeta_l(t)$  follows from definition of  $\mathcal{T}(H_o(s))$ . Boundedness of  $\zeta_{l_d}(t)$  follows from its definition in (24). Since  $\zeta_l(t)$  and  $\sigma(t)$  are bounded, it follows from (23) that  $u_l(t)$  is bounded, and (25) implies boundedness of  $\eta_l(t)$ . Notice that since  $u_l(t)$  and  $\epsilon_l(t)$  are bounded, it follows from (21) that  $\theta^\top x_l(t)$  is bounded. We notice that  $x_l(s)$  in (20) can be written as  $x_l(s) = H(s)(\theta^\top x_l(s) + \zeta_{l_d}(s))$ , which leads to boundedness of  $x_l(t)$ .  $\square$

For any  $\tau < \mathcal{T}(H_o(s))$  and any  $\epsilon_b > 0$ , Lemma 7 guarantees that the map  $\Delta_n : \mathbb{R}^+ \times [0, \mathcal{T}(H_o(s))] \rightarrow \mathbb{R}^+$

$$\Delta_n(\epsilon_b, \tau) = \max_{\|\epsilon_l\|_{\mathcal{L}_\infty} \leq \epsilon_b} \|\sigma + \eta_l\|_{\mathcal{L}_\infty} \quad (31)$$

is well defined. We note that strictly speaking  $\eta_l(t)$  depends not only on  $\epsilon_l(t)$  and  $\tau$ , but also upon other arguments, like  $\sigma(t)$  and other variables of the system that are used for definition of  $\eta_l(t)$ . These are dropped due to their non-crucial role in the subsequent analysis.

**Lemma 8:** Let  $\tau$  comply with (30), and  $\epsilon_b$  be any positive number. If  $\tilde{r}_l(t)$  is such that the resulting  $\epsilon_l(t)$  is bounded

$$\|\epsilon_l\|_{\mathcal{L}_\infty} \leq \epsilon_b, \quad (32)$$

and

$$2\omega\|u_l\|_{\mathcal{L}_\infty} + 2L\|x_l\|_{\mathcal{L}_\infty} + 2\Delta \geq \|\tilde{r}_l\|_{\mathcal{L}_\infty}, \quad (33)$$

where

$$\Delta = \Delta_n(\epsilon_b, \tau) + \delta_1, \quad (34)$$

$\delta_1$  is arbitrary positive constant, then  $\eta_l(t)$  is differentiable and the  $\mathcal{L}_\infty$  norm of  $\dot{\eta}_l(t)$  is finite.

**Proof:** It follows from (32) and Lemma 7 that  $x_l(t)$ ,  $u_l(t)$ ,  $\Delta_n(\epsilon_b, \tau)$  are bounded. Hence, it follows from (33) that  $\tilde{r}_l(t)$  is also bounded. Since  $C(s)$  is strictly proper and stable, bounded  $\tilde{r}_l(t)$  ensures that  $\epsilon_l(t)$  is differentiable with bounded derivative. Using similar methods, we prove that both  $u_l(t)$  and  $\zeta_{l_d}(t)$  have bounded derivatives. Since  $\dot{\sigma}(t)$  is bounded, it follows from (25) that  $\dot{\eta}_l(t)$  is bounded.  $\square$

For any  $\tau < \mathcal{T}(H_o(s))$  and any  $\epsilon_b > 0$ , Lemma 8 guarantees that the following map  $\Delta_d : \mathbb{R}^+ \times [0, \mathcal{T}(H_o(s))] \rightarrow \mathbb{R}^+$

$$\Delta_d(\epsilon_b, \tau) = \max_{\tilde{r}_l(t)} \|\dot{\sigma} + \dot{\eta}_l\|_{\mathcal{L}_\infty} \quad (35)$$

is well defined, where  $\tilde{r}_l(t)$  complies with (32) and (33). Further, let

$$\begin{aligned} \theta_m(\epsilon_b, \tau) &\triangleq \max_{\theta \in \Theta} \sum_{i=1}^n 4\theta_i^2 + 4\Delta^2 + 4(\omega_u - \omega_l)^2 \\ &\quad + 2\lambda_{\max}(P)\Delta_d(\epsilon_b, \tau)\Delta/\lambda_{\min}(Q), \end{aligned} \quad (36)$$

$$\epsilon_c(\epsilon_b, \tau) = \left\| C(s) \frac{1}{c_o^\top H(s)} c_o^\top \right\|_{\mathcal{L}_1} \sqrt{\frac{\theta_m(\epsilon_b, \tau)}{\lambda_{\max}(P)\epsilon_b^2}}. \quad (37)$$

We notice that for any finite  $\epsilon_b \in \mathbb{R}^+$  and any  $\tau$  verifying (30), we have finite  $\Delta_n(\epsilon_b, \tau)$  and  $\Delta_d(\epsilon_b, \tau)$ , and hence finite  $\epsilon_c(\epsilon_b, \tau)$ , if  $\tilde{r}_l(t)$  complies with (32) and (33).

#### 6.4 Time-delay Margin of the Closed-loop Adaptive System

In this section we analyze the time-delay margin for the closed-loop adaptive system with the  $\mathcal{L}_1$  adaptive controller. The main result is given by the following theorem.

**Theorem 2:** Consider the closed-loop adaptive system, comprised of System 1 in (12)-(14) with (3), (4)-(6), (8) and the LTI system in (20)-(23) in the presence of the same time delay  $\tau$ . For any  $\epsilon_b \in \mathbb{R}^+$  choose the set  $\Delta$  as in (34) and

$$\Gamma_c \geq \sqrt{\epsilon_c(\epsilon_b, \tau)} + \delta_2, \quad (38)$$

where  $\delta_2$  is arbitrary positive constant. Then for every  $\tau$  satisfying  $\tau < \mathcal{T}(H_o(s))$ , there exists exogenous signal  $\tilde{r}_l(t)$  ensuring  $\|\epsilon_l\|_{\mathcal{L}_\infty} < \epsilon_b$ , and

$$x_l(t) = x_d(t), \quad u_l(t) = u(t), \quad \forall t \geq 0. \quad (39)$$

The proof of this Theorem is given in the Appendix. Theorem 2 establishes the equivalence of state and control trajectories of the closed-loop adaptive system and the LTI system in (20)-(23) in the presence of the same time-delay. Therefore the time-delay margin of the system in (20)-(23) can be used as a conservative lower bound for the time-delay margin of the closed-loop adaptive system.

**Corollary 2:** Given the system in (1) and the  $\mathcal{L}_1$  adaptive controller defined via (3), (4)-(6) and (8) subject to (10), where  $\Gamma_c$  and  $\Delta$  are large enough, the closed-loop adaptive system is stable in the presence of time delay  $\tau$  in its output if  $\tau < \mathcal{T}(H_o(s))$ , where  $\mathcal{T}(H_o(s))$  is defined in (29).

The proof of Corollary 2 follows from Lemma 7 and Theorem 2 directly.

## 7 Gain Margin Analysis

We now analyze the gain margin of the system in (1) with  $\mathcal{L}_1$  adaptive controller. By inserting a gain module  $g$  into the control loop, the system in (1) can be formulated as:

$$\dot{x}(t) = A_m x(t) + b \left( \omega_g u(t) + \theta^\top(t) x(t) + \sigma(t) \right), \quad (40)$$

where  $\omega_g = g\omega$ . We note that this transformation implies that the set  $\Omega$  in the application of the Projection operator for adaptive laws needs to increase accordingly. However, increased  $\Omega$  will not violate the stability criterion. Thus, it follows from (7) that the gain margin of the  $\mathcal{L}_1$  adaptive controller is determined by:

$$\mathcal{G}_m = [\omega_l/\omega_{l_0}, \omega_u/\omega_{u_0}]. \quad (41)$$

If  $g \in \mathcal{G}_m$ , then the closed-loop system in (40) satisfies the  $\mathcal{L}_1$  stability criterion, implying that the entire closed-loop system is stable. We note that the lower-bound of  $\mathcal{G}_m$  is greater than zero. Eq. (41) implies that arbitrary gain margin can be obtained through appropriate choice of  $\Omega$ .

## 8 Main Results

Combining the results of Theorem 1, and Theorems of Sections 6.3 and 7, we have the following results:

**Theorem 3:** Given the system in (1) and the  $\mathcal{L}_1$  adaptive controller defined via (3), (4)-(6) and (8) subject to (10), we have:

$$\lim_{\Gamma_c \rightarrow \infty} (x(t) - x_{ref}(t)) = 0, \quad \forall t \geq 0, \quad (42)$$

$$\lim_{\Gamma_c \rightarrow \infty} (u(t) - u_{ref}(t)) = 0, \quad \forall t \geq 0, \quad (43)$$

$$\lim_{\Gamma_c \rightarrow \infty} \mathcal{T} \geq \mathcal{T}(H_o(s)), \quad \mathcal{G} \supseteq \mathcal{G}_m, \quad (44)$$

where  $\mathcal{T}$  and  $\mathcal{G}$  are the time-delay and gain margins of the  $\mathcal{L}_1$  adaptive controller, while  $\mathcal{T}(H_o(s))$ ,  $\mathcal{G}_m$  are defined in (29) and (41).

The inequalities in (44) imply that  $\mathcal{T}(H_o(s))$  and  $\mathcal{G}_m$  are just conservative bounds of the stability margins.

## 9 Simulations

We consider the same system from [1], in which a single-link robot arm is rotating on a vertical plane. Assuming constant  $\theta(t)$ , it can be cast into the form in (1) with  $A_m = \begin{bmatrix} 0 & 1 \\ -1 & -1.4 \end{bmatrix}$ ,  $b = \begin{bmatrix} 0 \\ 1 \end{bmatrix}$ ,  $c = \begin{bmatrix} 1 \\ 0 \end{bmatrix}$ . Let  $\theta = [2 \ 2]^\top$ ,  $\omega = 1$ ,  $\sigma(t) = \sin(\pi t)$ , so that the compact sets can be conservatively chosen as  $\Omega_0 = [0.2, 5]$ ,  $\Theta = [-10, 10]$ ,  $\Delta_0 = [-10, 10]$ , respectively. Next, we analyze the stability margins of the  $\mathcal{L}_1$  adaptive controller for this system numerically.

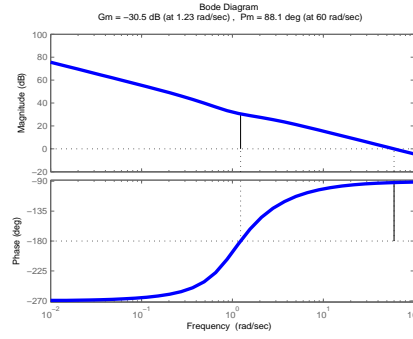
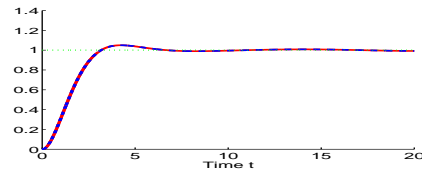


Fig. 4: Bode plot of  $H_o(s)$  for  $\theta = [2 \ 2]^\top$ ,  $\omega = 1$

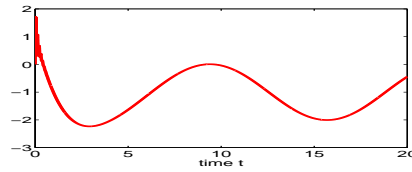
For  $\theta = [2 \ 2]^\top$ ,  $\omega = 1$  we can derive  $H_o(s)$  in (28) and look at its Bode plot in Fig. 4. It has phase margin  $88.1^\circ (1.54\text{rad})$  at cross frequency  $9.55\text{Hz} (60\text{rad/s})$ . Hence, the time-delay margin can be derived from (29) as:  $\mathcal{T}(H_o(s)) = \frac{1.54\text{rad}}{60\text{rad/s}} = 0.0256$ . We set  $\Delta = [-1000 \ 1000]^\top$ ,  $\Gamma_c = 500000$ , and run the  $\mathcal{L}_1$  adaptive controller with time-delay  $\tau = 0.02$ . The simulations in Figs. 5(a)-5(b) verify Corollary 2. As stated in Theorem 3, the time-delay margin of the LTI system in (28) provides only a conservative lower bound for the time-delay margin of the closed-loop adaptive system. So, we simulate the  $\mathcal{L}_1$  adaptive controller in the presence of larger time-delay, like  $\tau = 0.1$  sec., and observe that the system is not losing its stability. Since  $\theta$  and  $\omega$  are unknown to the controller, we derive the  $\mathcal{T}(H_o(s))$  for all possible  $\theta \in \Theta$  and  $\omega \in \Omega$  and use the most conservative value. It gives  $\mathcal{T}(H_o(s)) = 0.005s$ . The gain margin can be arbitrarily large as stated in (44).

## 10 Conclusion

In this paper, we derive the stability margins of  $\mathcal{L}_1$  adaptive controller presented in [1]. To the best of our knowledge, this is the first attempt to quantify the time-delay margin for general closed-loop adaptive systems. With the particular architecture presented in this paper, we prove that increasing the adaptive gain leads to improved transient tracking with improved stability margins. This presents a significant improvement over conventional adaptive control schemes, in which increasing the adaptive gain leads to reduced tolerance to time-delay in input/output channels.



(a)  $x_1(t)$  (solid),  $\hat{x}_1(t)$  (dashed), and  $r(t)$ (dotted)



(b) Time-history of  $u(t)$

Fig. 5: Performance of  $\mathcal{L}_1$  adaptive controller with time-delay  $0.02s$

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## Appendix

**Proof of Theorem 2:** In the closed-loop adaptive system in (15) for any  $t \geq 0$ , we notice that if  $\|(\sigma + \eta)_t\|_{\mathcal{L}_\infty} \leq \Delta$ , and  $\dot{\sigma}(t)$ ,  $\dot{\eta}(t)$  have finite derivatives over  $[0, t]$ , then application of  $\mathcal{L}_1$  adaptive controller from [1] is well-defined. Let  $d_t$  denote the truncated  $\mathcal{L}_\infty$  norm

$$d_t = \|(\dot{\sigma} + \dot{\eta})_t\|_{\mathcal{L}_\infty}. \quad (45)$$

It follows from (3) and (15) that  $\tilde{x}_q(s) = H(s)\tilde{r}(s)$ , where  $\tilde{x}_q(s)$  and  $\tilde{r}(s)$  are the Laplace transformations of  $\tilde{x}_q(t) = \hat{x}(t) - x_q(t)$  and

$$\tilde{r}(t) = \tilde{\omega}(t)u_q(t) + \tilde{\theta}^\top(t)x_q(t) + \tilde{\sigma}(t). \quad (46)$$

This along with Eq. (50) in [1] implies that

$$\begin{aligned} u_q(t)_{t \in [0, t]} &= \mathcal{S}\left(\frac{C(s)}{\omega}, u_q(0), (k_g r(t) - \theta^\top x_q(t) - \sigma(t) - \eta(t) - \tilde{r}(t))_{t \in [0, t]}\right), \\ \tilde{x}_q(t)_{t \in [0, t]} &= \mathcal{S}(H(s), \tilde{x}_q(0), \tilde{r}(t)_{t \in [0, t]}), \end{aligned} \quad (47)$$

where  $\tilde{\sigma}(t) = \hat{\sigma}(t) - (\sigma(t) + \eta(t))$ . Equation (47) implies that

$$\begin{aligned} u_q(t)_{t \in [0, t]} &= \mathcal{S}\left(\frac{C(s)}{\omega}, u_q(0), (k_g r(t) - \theta^\top x_q(t) - \sigma(t) - \eta(t))_{t \in [0, t]}\right) - \epsilon(t)_{t \in [0, t]}, \end{aligned} \quad (48)$$

where

$$\epsilon(t)_{t \in [0, t]} = \mathcal{S}(C(s)/\omega, 0, \tilde{r}(t)_{t \in [0, t]}). \quad (49)$$

We further define

$$\theta_t \triangleq \max_{\theta \in \Theta} \sum_{i=1}^n 4\theta_i^2 + \max_{\sigma \in \Delta} 4\sigma^2 + 4(\omega_u - \omega_l)^2 + 2\frac{\lambda_{\max}(P)}{\lambda_{\min}(Q)} d_t \Delta, \quad (50)$$

where  $d_t$  is defined in (45). It can be verified easily that Lemma 5 holds for truncated norms as well so that  $\|\tilde{x}_{q_t}\|_{\mathcal{L}_\infty} \leq \sqrt{\frac{\theta_t}{\lambda_{\min}(P)\Gamma_c}}$ . Since  $\epsilon(s) = \frac{C(s)}{\omega c_o^\top H(s)} c_o^\top H(s) \tilde{r}(s) = \frac{C(s)}{\omega c_o^\top H(s)} c_o^\top \tilde{x}_q(s)$ , then  $\epsilon(t)$  can be upper bounded as

$$\|\epsilon_t\|_{\mathcal{L}_\infty} \leq \left\| C(s) \frac{1}{\omega c_o^\top H(s)} c_o^\top \right\|_{\mathcal{L}_1} \sqrt{\frac{\theta_t}{\lambda_{\min}(P)\Gamma_c}}. \quad (51)$$

In the three steps below, we prove the existence of a continuously differentiable  $\eta(t)$  with uniformly bounded derivative in the closed-loop adaptive system (15), (3), (4)-(6), (8) and the existence of  $r_l(t)$  in time-delayed LTI system such that for any  $t \geq 0$ ,

$$|\sigma(t) + \eta(t)| < \Delta, x_o(t) = x_q(t), \quad (52)$$

$$\|\epsilon_{l_t}\|_{\mathcal{L}_\infty} < \epsilon_b, x_l(t) = x_q(t), u_l(t) = u_q(t), \epsilon_l(t) = \epsilon(t). \quad (53)$$

With (52), Lemma 6 implies that  $x_d(t) = x_q(t), u(t) = u_q(t)$  for any  $t \geq 0$ , which combining (53) proves Theorem 2.

Step 1: Let

$$\zeta(t) = \omega u_q(t) + \sigma(t). \quad (54)$$

We further define

$$\zeta_d = \begin{cases} 0, & t \in [0, \tau) \\ \zeta(t - \tau), & t \geq \tau \end{cases}. \quad (55)$$

Since (13) and (14) imply that  $x_o(t) = 0$  for any  $t \in [0, \tau]$ , it follows from (55) and the definition of the map  $\mathcal{S}$  that  $x_o(t)|_{t \in [0, \tau]} = \mathcal{S}(\bar{H}(s), x_o(0), \zeta_d(t)|_{t \in [0, \tau)})$ . For  $i \geq 1$ , it follows from the definition of the time-delayed open-loop system that

$$x_o(t)|_{t \in [i\tau, (i+1)\tau]} = \mathcal{S}(\bar{H}(s), x_o(i\tau), \zeta_d(t)|_{t \in [i\tau, (i+1)\tau)}) . \quad (56)$$

Hence, (56) holds for any  $i$ . We note that (49) implies that  $\epsilon(0) = 0$ . These along with (13), (14), (24), (26), imply that for  $i = 0$

$$\begin{aligned} u_q(i\tau) &= u_l(i\tau), \epsilon(i\tau) = \epsilon_l(i\tau), x_o(i\tau) = x_q(i\tau) = x_l(i\tau), \\ \zeta_d(t) &= \zeta_{ld}(t), t < (i+1)\tau, |\epsilon(t)| < \epsilon_b, \quad t \leq i\tau. \end{aligned}$$

Step 2: Assume that for any  $i$  the following conditions hold:

$$u_q(t) = u_l(t), \quad t \leq i\tau, \quad (57)$$

$$\epsilon(t) = \epsilon_l(t), \quad t = i\tau, \quad (58)$$

$$x_o(t) = x_q(t) = x_l(t), \quad t \leq i\tau, \quad (59)$$

$$\zeta_d(t) = \zeta_{ld}(t), \quad \forall t \in [i\tau, (i+1)\tau), \quad (60)$$

$$|\epsilon(t)| < \epsilon_b, \quad \forall t \leq i\tau. \quad (61)$$

For  $i \geq 1$ , further assume that there exist bounded  $\tilde{r}_l(t)$  and continuously differentiable  $\eta(t)$  with bounded derivative over  $t \in [0, i\tau)$  such that  $\forall t < i\tau$

$$\eta(t) = \eta_l(t), \quad |\sigma(t) + \eta(t)| < \Delta. \quad (62)$$

We prove below that there exist bounded  $\tilde{r}_l(t)$  and continuously differentiable  $\eta(t)$  with bounded derivative over  $t \in [0, (i+1)\tau]$  such that (57)-(62) hold for  $i+1$ , too.

We note that (20) implies that

$$x_l(t)|_{t \in [i\tau, (i+1)\tau]} = \mathcal{S}(\bar{H}(s), x_l(i\tau), \zeta_{l_d}(t)|_{t \in [i\tau, (i+1)\tau]}). \quad (63)$$

Using (59)-(60), it follows from (56) and (63) that

$$x_o(t) = x_l(t), \quad \forall t \in [i\tau, (i+1)\tau]. \quad (64)$$

We assumed in (62) that if  $i \geq 1$ , then there exists  $\eta(t)$  over  $[0, i\tau]$ . We now define  $\eta(t)$  over  $[i\tau, (i+1)\tau]$  as:

$$\eta(t) = \zeta_d(t) - \omega u_q(t) - \sigma(t), \quad t \in [i\tau, (i+1)\tau]. \quad (65)$$

Since (15) implies that  $x_q(t)|_{t \in [i\tau, (i+1)\tau]} = \mathcal{S}(\bar{H}(s), x_q(i\tau), (\omega u_q(t) + \sigma(t) + \eta(t))|_{t \in [i\tau, (i+1)\tau]})$ , it follows from (65) that  $x_q(t)|_{t \in [i\tau, (i+1)\tau]} = \mathcal{S}(\bar{H}(s), x_q(i\tau), \zeta_d(t)|_{t \in [i\tau, (i+1)\tau]})$ . Along with (56) and (59) this ensures that

$$x_q(t) = x_o(t), \quad \forall t \in [i\tau, (i+1)\tau]. \quad (66)$$

However, the definition in (65) does not guarantee

$$|\sigma(t) + \eta(t)| < \Delta, \quad t \in [i\tau, (i+1)\tau], \quad (67)$$

which is required for application of  $\mathcal{L}_1$  adaptive controller.

We prove (67) by contradiction. Since  $\eta(t)$  is continuous over  $[i\tau, (i+1)\tau]$ , if (67) is not true, there must exist  $t' \in [i\tau, (i+1)\tau]$  such that

$$|\sigma(t) + \eta(t)| < \Delta, \quad \forall t < t', \quad (68)$$

$$|\sigma(t') + \eta(t')| = \Delta. \quad (69)$$

It follows from (56) and (65) that  $x_o(t)|_{t \in [i\tau, t']} = \mathcal{S}(\bar{H}(s), x_o(i\tau), (\omega u_q(t) + \sigma(t) + \eta(t))|_{t \in [i\tau, t']})$ . It follows from (48) and (49) that

$$\begin{aligned} u_q(t)|_{t \in [i\tau, t']} &= \mathcal{S}\left(\frac{C(s)}{\omega}, u_q(i\tau) + \epsilon(i\tau), (k_g r(t) - \theta^\top x_q(t) \right. \\ &\quad \left. - \sigma(t) - \eta(t))|_{t \in [i\tau, t']}\right) - \epsilon(t)|_{t \in [i\tau, t']}, \end{aligned} \quad (70)$$

where

$$\epsilon(t)|_{t \in [i\tau, t']} = \mathcal{S}\left(C(s)/\omega, \epsilon(i\tau), \tilde{r}(t)|_{t \in [i\tau, t']}\right). \quad (71)$$

We notice that if  $i \geq 1$ , then on  $[0, i\tau]$  we have  $\tilde{r}_l(t)$  well defined. Let

$$\tilde{r}_l(t) = \tilde{r}(t), \quad t \in [i\tau, t']. \quad (72)$$

We have  $\epsilon_l|_{t \in [i\tau, t']} = \mathcal{S}\left(C(s)/\omega, \epsilon_l(i\tau), \tilde{r}(t)_{t \in [i\tau, t']}\right)$ , which along with (58) and (71) imply that

$$\epsilon_l(t) = \epsilon(t), \quad \forall t \in [i\tau, t']. \quad (73)$$

Hence, (57), (64), (66), (70) yield

$$\begin{aligned} u_q(t)|_{t \in [i\tau, t']} &= \mathcal{S}\left(C(s)/\omega, u_l(i\tau) + \epsilon(i\tau), (k_g r(t) \right. \\ &\quad \left. - \theta^\top x_l(t) - \sigma(t) - \eta(t))_{t \in [i\tau, t']}\right) - \epsilon(t)|_{t \in [i\tau, t']}. \end{aligned} \quad (74)$$

It follows from (73) and (74) that

$$\begin{aligned} u_q(t)|_{t \in [i\tau, t']} &= \mathcal{S}\left(C(s)/\omega, u_l(i\tau) + \epsilon_l(i\tau), (k_g r(t) \right. \\ &\quad \left. - \theta^\top x_l(t) - \sigma(t) - \eta(t))_{t \in [i\tau, t']}\right) - \epsilon_l(t)|_{t \in [i\tau, t']}. \end{aligned} \quad (75)$$

It follows from (25) and (60) that

$$\eta_l(t) = \zeta_d(t) - \omega u_l(t) - \sigma(t), \quad t \in [i\tau, t'], \quad (76)$$

which along with (21) yields

$$\begin{aligned} u_l(t)|_{t \in [i\tau, t']} &= \mathcal{S}\left(\frac{C(s)}{\omega}, u_l(i\tau) + \epsilon_l(i\tau), (k_g r(t) \right. \\ &\quad \left. - \theta^\top x_l(t) - \sigma(t) - \eta_l(t))_{t \in [i\tau, t']}\right) - \epsilon_l(t)|_{t \in [i\tau, t']}. \end{aligned} \quad (77)$$

From (65), (75), (76) and (77), we have

$$u_q(t) = u_l(t), \quad \forall t \in [i\tau, t'] \quad (78)$$

$$\eta(t) = \eta_l(t), \quad \forall t \in [i\tau, t']. \quad (79)$$

It follows from (62) and (79) that

$$\eta(t) = \eta_l(t), \quad \forall t \in [0, t']. \quad (80)$$

We now prove by contradiction that

$$|\epsilon(t)| < \epsilon_b, \quad \forall t \in [i\tau, t']. \quad (81)$$

If (81) is not true, then since  $\epsilon(t)$  is continuous, there exists some  $\bar{t} \in [i\tau, t']$  such that

$$|\epsilon(t)| < \epsilon_b, \quad \forall t \in [i\tau, \bar{t}], \quad (82)$$

$$|\epsilon(\bar{t})| = \epsilon_b. \quad (83)$$

It follows from (61) that

$$|\epsilon(t)| \leq \epsilon_b, \quad \forall [0, \bar{t}]. \quad (84)$$

It follows from (57), (59), (64), (66) and (78) that  $u_q(t) = u_l(t)$ ,  $x_q(t) = x_l(t)$  for any  $t \in [0, \bar{t}]$ . Therefore, (46) and (72) imply that  $\tilde{r}_l(t) = \tilde{\omega}(t)u_l(t) + \tilde{\theta}^\top(t)x_l(t) + \tilde{\sigma}(t)$ , and hence

$$\|\tilde{r}_{l\bar{t}}\|_{\mathcal{L}_\infty} \leq 2\omega\|u_{l\bar{t}}\|_{\mathcal{L}_\infty} + L\|x_{l\bar{t}}\|_{\mathcal{L}_\infty} + 2\Delta. \quad (85)$$

From (84) and (85), Lemmas 7 and 8 imply that  $\eta_l(t)$  is bounded and differentiable with bounded derivative. Further, it follows from (31) and (35) that

$$\begin{aligned} |\sigma(t) + \eta_l(t)| &\leq \Delta_n(\epsilon_b, \tau), \quad \forall t \in [0, \bar{t}], \\ |\dot{\sigma}(t) + \dot{\eta}_l(t)| &\leq \Delta_d(\epsilon_b, \tau), \quad \forall t \in [0, \bar{t}]. \end{aligned} \quad (86)$$

Since (80) holds,  $\eta(t)$  is also bounded and differentiable with bounded derivative over  $[0, t']$  and further

$$|\sigma(t) + \eta(t)| \leq \Delta_n(\epsilon_b, \tau), \quad \forall t \in [0, \bar{t}], \quad (87)$$

$$|\dot{\sigma}(t) + \dot{\eta}(t)| \leq \Delta_d(\epsilon_b, \tau), \quad \forall t \in [0, \bar{t}]. \quad (88)$$

It follows from (51) that

$$\|\epsilon_{\bar{t}}\|_{\mathcal{L}_\infty} \leq \left\| C(s) \frac{1}{\omega c_o^\top H(s)} c_o^\top \right\|_{\mathcal{L}_1} \sqrt{\frac{\theta_{\bar{t}}}{\lambda_{\min}(P)\Gamma_c}}. \quad (89)$$

It follows from (36), (50) and (88) that

$$\theta_{\bar{t}} \leq \theta_m(\epsilon_b, \tau). \quad (90)$$

Hence, (51) and (90) imply that  $\|\epsilon_{\bar{t}}\|_{\mathcal{L}_\infty} \leq \left\| C(s) \frac{1}{\omega c_o^\top H(s)} c_o^\top \right\|_{\mathcal{L}_1} \sqrt{\frac{\theta_m(\epsilon_b, \tau)}{\lambda_{\min}(P)\Gamma_c}}$ . From (37) and (38) we have  $\|\epsilon_{\bar{t}}\|_{\mathcal{L}_\infty} < \epsilon_b$ , which contradicts (83). Therefore, (81) holds.

If (81) is true, it follows from (61) that

$$|\epsilon(t)| < \epsilon_b, \quad \forall t \in [0, t'].$$

Hence, it follows from (31) and (80) that

$$|\sigma(t) + \eta(t)| \leq \Delta_n < \Delta, \quad (91)$$

which contradicts (69). Hence, we have

$$|\sigma(t) + \eta(t)| < \Delta, \quad \forall t \in [i\tau, (i+1)\tau]. \quad (92)$$

Therefore, combining (64), (66), (73), (78), (79), (81), (92), we proved that there exist  $\tilde{r}_l(t)$  and continuously differentiable  $\eta(t)$  in  $[0, (i+1)\tau)$ , which ensures

$$x_o(t) = x_q(t) = x_l(t), \quad \forall t \in [i\tau, (i+1)\tau], \quad (93)$$

$$\epsilon(t) = \epsilon_l(t), \quad \forall t \in [i\tau, (i+1)\tau], \quad (94)$$

$$u_q(t) = u_l(t), \quad \forall t \in [i\tau, (i+1)\tau], \quad (95)$$

$$\eta(t) = \eta_l(t), \quad \forall t \in [i\tau, (i+1)\tau], \quad (96)$$

$$|\epsilon(t)| < \epsilon_b, \quad \forall t \in [0, (i+1)\tau], \quad (97)$$

$$|\sigma(t) + \eta(t)| < \Delta, \quad \forall t \in [0, (i+1)\tau]. \quad (98)$$

It follows from (23), (54) and (95) that

$$\zeta(t) = \zeta_l(t), \quad \forall t \in [i\tau, (i+1)\tau).$$

Therefore (24) and (55) imply that

$$\zeta_d(t) = \zeta_{l_d}(t), \quad \forall t \in [(i+1)\tau, (i+2)\tau). \quad (99)$$

We note that Step 2 is proved in (93)-(99) for  $i+1$ .

Step 3: Step 1 implies that the relationships (57)-(61) hold for  $i=0$ . By iterating the results from Step 2, we prove (52)-(53), which conclude proof of the Theorem.  $\square$